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LETTER TO THE EDITOR

Quantum integrable systems and Clebsch–Gordan series. I

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Abstract. The class of quantum integrable systems associated with root systems was introduced as a generalization of the Calogero–Sutherland systems. In this letter, a new property of such systems is proved to be valid. Namely, in the case of the potential $v(q) = \sin^{-2} q$, the series for the product of two wavefunctions coincides with the Clebsch–Gordan series. This gives the recursive relations for the wavefunctions of such systems and for generalized spherical functions related to them on symmetric spaces.

One conjectures that the Clebsch–Gordan series is also unchanged under more general two-parametric deformation ((q, t)-deformation).

1. Introduction

The class of quantum integrable systems associated with root systems was introduced in [8] (see also [9]) as the generalization of the Calogero–Sutherland systems [1, 14]. Such systems depend on one real parameter κ (the type A–D–E), on two parameters (the type B_n, C_n, F_4 and G_2) and on three parameters for the type BC_n. These parameters are related to the coupling constants of the quantum system.

The change of parameters defines a deformation of Weyl formulae [18] for characters of the compact simple Lie groups ($\kappa = 1$) and correspondingly for zonal spherical functions on symmetric spaces [2, 3] (at special values of κ, for example $\kappa = \frac{1}{2}, 2, 4$).

This class has many remarkable properties. We only mention that the wavefunctions of such systems are a natural generalization of special functions (hypergeometric functions) for the case of several variables. The history of the problem and some results can be found in [10]. Here we shall consider another such property: the product of two wavefunctions is a finite linear combination of analogous functions, namely of functions which appeared in the corresponding Clebsch–Gordan series. In other words, this deformation (κ-deformation) does not change the Clebsch–Gordan series. For the rank 1, we get the well-known cases of the Legendre, Gegenbauer and Jacobi polynomials and the limiting cases of the Laguerre and Hermite polynomials (see for example [15]). Some other cases were considered more recently in [5, 16, 12, 17, 7, 6, 13]. Note that this approach not only gives new results, but also a new insight into some old ones.

We conjecture that these results remain valid for the more general (q, t)-deformation introduced by Macdonald [7].

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2. General description

The systems under consideration are described by the Hamiltonian (for more details see [10])

$$H = \tfrac{1}{2}p^2 + U(q) \qquad p^2 = (p, p) = \sum_{j=1}^{l} p_j^2 \tag{2.1}$$

where $p = (p_1, \ldots, p_l)$, $p_j = -\mathrm{i}\frac{\partial}{\partial q_j}$, is a momentum vector operator, and $q = (q_1, \ldots, q_l)$ is a coordinate one in the l-dimensional vector space $V \sim \mathbb{R}^l$ with the standard scalar product (α, q). They are a generalization of the Calogero–Sutherland systems [1, 14] for which $\{\alpha\} = \{e_i - e_j\}$, $\{e_j\}$ is a standard basis in V. The potential $U(q)$ is constructed by means of the certain system of vectors $R^+ = \{\alpha\}$ in V (the so-called root system):

$$U = \sum_{\alpha\in R^+} g_\alpha^2 v(q_\alpha) \qquad q_\alpha = (\alpha, q) \qquad g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1). \tag{2.2}$$

The constants satisfy the condition $g_\alpha = g_\beta$, if $(\alpha, \alpha) = (\beta, \beta)$. Such systems are completely integrable for $v(q)$ of five types. Here we only consider the case of $v(q) = \sin^{-2} q$.

3. Root systems

We give here only basic definitions. For more details see [4, 7, 11].

Let V be a l-dimensional real vector space with a standard scalar product (,), $(\alpha, \beta) = \sum \alpha_j\beta_j$, and let s_α be the reflection in the hyperplane through the origin orthogonal to the vector α

$$s_\alpha q = q - (q, \alpha^\vee)\alpha \qquad \alpha^\vee = (2/(\alpha, \alpha))\alpha. \tag{3.1}$$

Consider a finite set of nonzero vectors $R = \{\alpha\}$ generating V and satisfying the following conditions:

(1) for any $\alpha \in R$, the reflection s_α conserves R: $s_\alpha R = R$;

(2) for all $\alpha, \beta \in R$, we have $(\alpha^\vee, \beta) \in \mathbb{Z}$.

The set $\{s_\alpha\}$ generates the finite group $W(R)$ (the Weyl group of R). The root system R is called a reduced one if only vectors in R collinear to α are $\pm\alpha$. Let us choose the hyperplane which does not contain the root. Then the root system $R = R^+ \bigcup R^-$, and R^+ is the set of positive roots. In R^+ there is the basis (simple roots) $\{\alpha_1, \ldots, \alpha_l\}$ such that any $\alpha \in R^+$, $\alpha = \sum_j n_j\alpha_j$, $n_j \geqslant 0$. The root system R is called irreducible if it cannot be union of two nonempty subsets R_1 and R_2 which are orthogonal to each other.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots in R, R^+ be the set of positive roots and $\{\lambda_j\}$ be a dual basis or the weight basis: $(\lambda_j, \alpha_k) = \delta_{jk}$.

Let Q be the root lattice and Q^+ be the cone of positive roots

$$Q = \left\{\beta \colon \beta = \sum_{j=1}^{l} m_j\alpha_j, m_j \in \mathbb{Z}\right\} \qquad Q^+ = \left\{\gamma \colon \gamma = \sum_{j=1}^{l} n_j\alpha_j, n_j \in \mathbb{N}\right\}. \tag{3.2}$$

Let P be the weight lattice and P^+ be the cone of dominant weights:

$$P = \left\{\lambda \colon \lambda = \sum_{j=1}^{l} m_j\lambda_j, m_j \in \mathbb{Z}\right\} \qquad P^+ = \left\{\mu \colon \mu = \sum_{j=1}^{l} n_j\lambda_j, n_j \in \mathbb{N}\right\}. \tag{3.3}$$

According to [16, 7] we define a partial order on P as follows $\lambda \geqslant \mu$ if and only if $\lambda - \mu \in Q^+$ (or $(\lambda, \lambda_j) \geqslant (\mu, \lambda_j)$ for all $j = 1, \ldots, l$). The set of linear combinations over

$\mathbb{R}$ of the functions $f_\lambda(q) = \exp\{2\mathrm{i}(\lambda, q)\}$, $\lambda \in P$, $q \in V$ may be considered as the group algebra A over $\mathbb{R}$ of the free Abelian group P. For any $\lambda \in P$, let $\mathrm{e}^\lambda \sim f_\lambda(q)$ denote the corresponding element of A, so that $\mathrm{e}^\lambda \mathrm{e}^\mu = \mathrm{e}^{\lambda+\mu}$, $(\mathrm{e}^\lambda)^{-1} = \mathrm{e}^{-\lambda}$ and $\mathrm{e}^0 = 1$, the identity element of A. Then e^λ, $\lambda \in P$ form an $\mathbb{R}$-basis of A.

The Weyl group $W(R)$ acts on P and hence also on $A : s(\mathrm{e}^\lambda) = \mathrm{e}^{s\lambda}$ for $s \in W$ and $\lambda \in P$. Let A^W denote the subalgebra of W-invariant elements of A. Since each W-orbit in P contains exactly one point in P^+, the monomial symmetric functions

$$m_\lambda = \sum_{\mu \in W\cdot\lambda} \mathrm{e}^\mu \qquad \lambda \in P^+ \tag{3.4}$$

form an $\mathbb{R}$-basis of A^W.

4. Clebsch–Gordan series

The Schrödinger equation for the quantum system related to the root system R with $v(q_\alpha) = \sin^{-2} q_\alpha$, $q_\alpha = (q, \alpha)$ has the form

$$H\Psi^\kappa = E(\kappa)\Psi^\kappa \qquad H = -\Delta_2 + \sum_{\alpha\in R^+} \kappa_\alpha(\kappa_\alpha - 1)\sin^{-2} q_\alpha \qquad \Delta_2 = \sum_{j=1}^{l} \frac{\partial^2}{\partial q_j^2} \tag{4.1}$$

and for the ground-state wavefunction we have

$$\Psi_0^\kappa(q) = \prod_{\alpha\in R^+} (\sin q_\alpha)^{\kappa_\alpha} \qquad \kappa_\alpha = \kappa_\beta \qquad \text{if } (\alpha,\alpha) = (\beta,\beta)$$
$$E_0(\kappa) = (\gamma,\gamma) \qquad \gamma = \sum_{\alpha\in R^+} \kappa_\alpha \alpha. \tag{4.2}$$

Substituting $\Psi_\lambda^\kappa = \Phi_\lambda^\kappa \Psi_0^\kappa$ we obtain

$$-\Delta^\kappa \Phi_\lambda^\kappa = \varepsilon_\lambda(\kappa)\Phi_\lambda^\kappa \qquad \Delta^\kappa = \Delta_2 + \Delta_1^\kappa \qquad \varepsilon_\lambda(\kappa) = E_\lambda(\kappa) - E_0(\kappa). \tag{4.3}$$

Here the operator Δ_1^κ takes the form

$$\Delta_1^\kappa = \tfrac{1}{2}\sum_{\alpha\in R^+} |\alpha|^2 \kappa_\alpha (\cot q_\alpha)\partial_\alpha \qquad \partial_\alpha = (\alpha, \partial) \qquad |\alpha|^2 = (\alpha,\alpha). \tag{4.4}$$

It is easy to see that Δ^κ maps A^W into A^W and

$$\Delta^\kappa m_\lambda = (\lambda + 2\rho(\kappa), \lambda) m_\lambda + \text{lower terms} \qquad 2\rho(\kappa) = \sum_\alpha \kappa_\alpha \alpha, \alpha \in R^+. \tag{4.5}$$

So $\varepsilon_\lambda(\kappa) = (\lambda + 2\rho(\kappa), \lambda)$ and $\Phi_\lambda^\kappa = \sum_{\mu\leqslant\lambda} C_\lambda^\mu(\kappa) m_\mu \qquad \mu \in P^+.$

From this,

$$\Phi_\lambda^\kappa \Phi_\mu^\kappa = \sum_{\nu\leqslant\lambda+\mu} C_{\lambda\mu}^\nu(\kappa)\Phi_\nu^\kappa \qquad \nu \in P^+. \tag{4.6}$$

In fact, we could obtain a stronger condition by using the orthogonality properties of $\Phi_\lambda^\kappa(q)$ which follow from the self-adjointness of the operator H

$$\int \bar{\Phi}_\lambda^\kappa(q)\Phi_\nu^\kappa(q)\,\mathrm{d}\mu(q) = 0 \qquad \text{if } \lambda \neq \mu \qquad \mathrm{d}\mu\ \ (q) = |\Psi_0^\kappa(q)|^2\,\mathrm{d}^l q. \tag{4.7}$$

Namely,

$$C_{\mu\lambda}^\nu(\kappa) = 0 \qquad \text{if } \int \bar{\Phi}_\nu^\kappa \Phi_\lambda^\kappa \Phi_\mu^\kappa\,\mathrm{d}\mu(q) = 0. \tag{4.8}$$

But $\bar{\Phi}^{\kappa}_{\nu}$ is the eigenfunction of Δ^{κ} and hence should have the form $\Phi^{\kappa}_{\tilde{\nu}}$. So

$$\Phi^{\kappa}_{\mu}\Phi^{\kappa}_{\lambda} = \sum_{\lambda-\tilde{\mu}\leqslant\nu\leqslant\lambda+\mu} C^{\nu}_{\mu\lambda}(\kappa)\Phi^{\kappa}_{\nu} \qquad \nu \in P^{+} \tag{4.9}$$

or

$$\Phi^{\kappa}_{\mu}(q)\Phi^{\kappa}_{\lambda}(q) = \sum_{\nu\in D_1(\mu,\lambda)} C^{\nu}_{\mu\lambda}(\kappa)\Phi^{\kappa}_{\nu}(q) \qquad \nu \in P^{+}. \tag{4.10}$$

Here $D_1(\mu, \lambda)$ is the set of ν defined by conditions

$$\lambda + \mu_N \leqslant \nu \leqslant \lambda + \mu_1 \qquad \nu \in P^{+} \tag{4.11}$$

$\mu_1 = \mu$ is the highest weight and μ_N is the lowest weight of the weight diagram D_μ defined by μ.

Note that in the case when all $\nu \in P^{+}$

$$D_1(\mu, \lambda) \supset D_\mu(\lambda) \tag{4.12}$$

where $D_\mu(\lambda)$ is defined by the formula

$$D_\mu(\lambda) = (D_\mu + \lambda) \cap P^{+}. \tag{4.13}$$

Here D_μ is the weight diagram defined by the dominant weight $\mu = \mu_1$.

As it was shown in [16] the set of ν in (4.9) for all $s \in W$ should satisfy the condition

$$\nu = \lambda + \tilde{\mu} \qquad \lambda + \mu_N \leqslant \lambda + s\tilde{\mu} \leqslant \lambda + \mu_1 \qquad s \in W. \tag{4.14}$$

Lemma. Let the weight σ satisfies the condition

$$\mu_N \leqslant s\sigma \leqslant \mu_1 \tag{4.15}$$

for all $s \in W$. Then σ belongs to the weight diagram D_μ, $\mu = \mu_1$.

Proof. It is evident that if $\sigma \in D_\mu$, then $s\sigma$ also belongs to D_μ and condition (4.15) is satisfied. Let us suppose now that $\sigma \notin D_\mu$, but

$$\mu_N \leqslant \sigma \leqslant \mu_1$$

and consider the set $\mathcal{O}_\sigma = \{\sigma_j\} = \{s_j\sigma : s_j \in W\}$. Let σ_1 be the highest weight and σ_M be the lowest weight in $\mathcal{O}_\sigma$. Then $\sigma_1 = s_1\sigma \in P^{+}$ and $\sigma_1 > \mu_1$; $\sigma_M = s_M\sigma$ and $\sigma_M < \mu_N$. In other words, in this case σ_1 defines the weight diagram D_{σ_1} such that $\sigma_1 > \mu_1$ which contradicts (4.15). □

Theorem 1. The κ-deformed Clebsch–Gordan series has the form

$$\Phi^{\kappa}_{\mu}\Phi^{\kappa}_{\lambda} = \sum_{\nu\in D_\mu(\lambda)} C^{\nu}_{\mu\lambda}(\kappa)\Phi^{\kappa}_{\nu} \tag{4.16}$$

or

$$\Phi^{\kappa}_{\mu}\Phi^{\kappa}_{\lambda} = \sum_{\nu\in D_\lambda(\mu)} C^{\nu}_{\mu\lambda}(\kappa)\Phi^{\kappa}_{\nu}. \tag{4.17}$$

5. A_1 case

In this case, the representation is characterized by the integer non-negative number l, and we have the differential equation for Φ_l^κ

$$-((\Phi_l^\kappa)'' + 2\kappa \cot x (\Phi_l^\kappa)') = \varepsilon_l(\kappa)\Phi_l^\kappa \qquad f' = \mathrm{d}f/\mathrm{d}x. \tag{5.1}$$

The solution normalized by the condition $\Phi_l^\kappa(0) = 1$ has the form

$$\Phi_l^\kappa(x) = c_l(\kappa)P_l^\kappa(z) \qquad z = 2\cos x \qquad P_l^\kappa \sim z^l \qquad \text{at } z \to \infty \tag{5.2}$$

where $P_l^\kappa(z) \sim C_l^\kappa(z/2)$ and $C_l^\kappa(z)$ is the Gegenbauer polynomial.

The κ-deformed Clebsch–Gordan series now takes the form

$$P_m^\kappa(z)P_n^\kappa(z) = \sum_{l=|m-n|}^{m+n} C_{mn}^l(\kappa)P_l^\kappa(z) \tag{5.3}$$

where l has the same parity as $(m+n)$. The coefficients $C_{mn}^l(\kappa)$ may be calculated explicitly. For the simplest case we have

$$zP_n^\kappa(z) = P_{n+1}^\kappa(z) + a_n(\kappa)P_{n-1}^\kappa(z) \qquad a_n(\kappa) = \frac{n(n-1+2\kappa)}{(n-1+\kappa)(n+\kappa)}. \tag{5.4}$$

The quantity $d_n(\kappa) = c_n^{-1}(\kappa) = P_n^\kappa(2)$ may be considered as a κ-deformed dimension of irreducible representations of the Lie algebra A_1. We have $d_{n+1}(\kappa) = 2d_n(\kappa) - a_n(\kappa)d_{n-1}(\kappa)$, $d_0 = 1, d_1 = 2$. From this, we obtain $d_n(\kappa) = (2\kappa)_n/(\kappa)_n$, where $(\kappa)_n = (\kappa)(\kappa+1)\ldots(\kappa+n-1)$, $(\kappa)_0 = 1$.

6. A_2 case

In this case, the function Φ_μ^κ is determined by two integer non-negative numbers m and n: $\mu = m\lambda_1 + n\lambda_2$, where λ_1 and λ_2 are two fundamental weights. Also it is a solution of (4.3) with $\varepsilon_{mn} = m^2 + n^2 + mn + 3\kappa(m+n)$, $\Phi_{mn}^\kappa = c_{mn}(\kappa)P_{mn}^\kappa$; $P_{mn}^\kappa = m_\mu +$ lower terms. The Clebsch–Gordan series for $P_\mu^\kappa(z_1, z_2)$ is given by formula (4.16).

Let us give an example for $P_\mu^\kappa = P_{1,0}^\kappa = z_1$:

$$z_1 P_{m,n}^\kappa = P_{m+1,n}^\kappa + a_{m,n}(\kappa)P_{m,n-1}^\kappa + b_{m,n}(\kappa)P_{m-1,n+1}^\kappa \tag{6.1}$$

where $a_{m,n}(\kappa), b_{m,n}(\kappa)$ are rational functions of κ. The formula for $P_{01}^\kappa P_{mn}^\kappa$ is analogous. Note that $P_{1,0}^\kappa \equiv P_{1,0}^1$, $P_{0,1}^\kappa \equiv P_{0,1}^1$, i.e. do not depend on κ.

So we could express $P_{m,n+1}^\kappa$ through P_{kl}^κ for $l \leqslant n$, and $P_{m+1,n}^\kappa$ through P_{kl}^κ for $k \leqslant m$, correspondingly.

To find the coefficients $a_{mn}(\kappa), b_{mn}(\kappa)$ we consider equation (4.3) in new variables z_1 and z_2 which are characters of two fundamental representations of A_2

$$z_1 = \exp(2\mathrm{i}q_1) + \exp(2\mathrm{i}q_2) + \exp(2\mathrm{i}q_3) \qquad z_2 = \bar{z}_1. \tag{6.2}$$

Denoting the derivatives $\partial_1 = \partial/\partial z_1$, $\partial_2 = \partial/\partial z_2$, we have

$$-\Delta^\kappa = (z_1^2 - 3z_2)\partial_1^2 + (z_2^2 - 3z_1)\partial_2^2 + (z_1 z_2 - 9)\partial_1\partial_2 + (3\kappa+1)(z_1\partial_1 + z_2\partial_2). \tag{6.3}$$

Note that Δ^κ is self-adjoint in the space of functions $f(z, \bar{z})$ with the norm [5]

$$\|f\|_\kappa^2 = \int_D |f(z,\bar{z})|^2 (w(z,\bar{z}))^\kappa \,\mathrm{d}z\,\mathrm{d}\bar{z} \qquad \kappa > -\tfrac{1}{3}$$
$$w(z,\bar{z}) = -z^2\bar{z}^2 + 4z^3 + 4\bar{z}^3 - 18z\bar{z} + 27$$

where D is a bounded domain defined by the curve $w(z, \bar{z}) = 0$.

The polynomial $P^{\kappa}_{pq}(z_1, z_2)$ has the form

$$P^{\kappa}_{pq} = \sum_{mn} C^{pq}_{mn}(\kappa) z_1^m z_2^n \qquad m+n \leqslant p+q \qquad m-n \equiv p-q (\mathrm{mod}\ 3). \tag{6.4}$$

From (4.3) and (6.3) one can find a few first coefficients of $P^{\kappa}_{pq}(z_1, z_2)$

$$C^{p,q}_{p+1,q-2}(\kappa) = -\frac{q(q-1)}{\kappa+q-1} \qquad C^{p,q}_{p-2,q+1}(\kappa) = -\frac{p(p-1)}{\kappa+p-1} \tag{6.5}$$

$$C^{pq}_{p-1,q-1}(\kappa) = -\frac{pq(3\kappa^2+\alpha\kappa+\beta)}{(\kappa+p-1)(\kappa+q-1)(2\kappa+p+q-1)}. \tag{6.6}$$

Here $\alpha = -[2pq - 3(p+q) + 4]$ and $\beta = -(p+q-1)(p-1)(q-1)$.

By using (6.5) and (6.6) we obtain the explicit expression for coefficients $a_{mn}(\kappa)$ and $b_{mn}(\kappa)$ in (6.1)

$$a_{mn} = \frac{n(n+m+\kappa)(n-1+2\kappa)(n+m-1+3\kappa)}{(n+\kappa)(n+m+2\kappa)(n-1+\kappa)(n+m-1+2\kappa)} \tag{6.7}$$

$$b_{mn} = \frac{m(m-1+2\kappa)}{(m+\kappa)(m-1+\kappa)} \tag{6.8}$$

and than the recursive formula for $d_{mn}(\kappa)$:

$$3d_{mn}(\kappa) = d_{m+1,n}(\kappa) + a_{m,n}(\kappa) d_{m,n-1}(\kappa) + b_{m,n}(\kappa) d_{m-1,n+1}(\kappa). \tag{6.9}$$

We have also

$$d_{0,0}(\kappa) = 1 \qquad d_{1,0}(\kappa) = d_{0,1}(\kappa) = 3 \qquad d_{1,1}(\kappa) = 6\frac{3\kappa+1}{2\kappa+1}. \tag{6.10}$$

Solving the recursive relation (6.9) with the initial condition (6.10) we obtain the explicit expression for $d_{mn}(\kappa)$ which is the κ-deformed Weyl formula for the dimension of irreducible representations of the Lie algebra A_2

$$d_{mn}(\kappa) = \frac{(2\kappa)_m (2\kappa)_n (3\kappa)_{m+n}}{(\kappa)_m (\kappa)_n (2\kappa)_{m+n}} \qquad (\kappa)_n = \kappa(\kappa+1)\ldots(\kappa+n-1). \tag{6.11}$$

A more detailed version of this letter will be published elsewhere.

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